def: The movement of the MC (Markov chain) from 1 state to another is called a transition.

... move in ONE STEP

def: Transition probability : $P(X_{n+1} = j \mid X_n = i)$ = P_{ij} (denoted)

$\Rightarrow$ probability of moving from state i (at current time n) to state j (in the next time step, $n+1$)

* In many examples, the transition prob. does NOT depend on time n $\rightarrow$ this is a stationary transition prob.

$$\boxed{P(X_{n+1} = j \mid X_n = i) = P(X_1 = j \mid X_0 = i)}$$

Arrange these P_{ij}'s in a matrix :

def: $\mathbb{P}$ - matrix of 1-step transition probabilities

Let $S = \{0, 1, 2, \dots\}$ - state space

$$\mathbb{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ i \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & \cdots & j & \cdots \end{array} \\ \begin{bmatrix} P_{00} & P_{01} & P_{12} & \cdots & P_{0j} & \cdots \\ P_{10} & P_{11} & P_{12} & \cdots & P_{1j} & \\ P_{20} & P_{21} & P_{22} & & \vdots & \\ \vdots & \vdots & \vdots & & \vdots & \ddots \\ P_{i0} & P_{i1} & P_{i2} & \cdots & P_{ij} & \cdots \\ & \vdots & & \ddots & \vdots & \end{bmatrix} \end{array}$$

$\leftarrow$ future states

current states

$P_{ij} \geq 0 \ \forall i,j$, $\displaystyle\sum_{j=0}^{\infty} P_{ij} = 1$ for $i = 0, 1, \dots$ $\leftarrow$ Rows Σ to 1 ✓

Example 1: Forecasting the Weather

Suppose that the chance of rain tomorrow depends only on whether or not it is raining today:

- If it rains today, then it will rain tomorrow w/prob. α
- If it doesn't rain today, then it will rain tomorrow w/prob. β

Assume process is defined as
- state 0 when it rains
- state 1 when it doesn't rain

$$\begin{array}{c|cc} & 0 & 1 \\ \hline \end{array} \leftarrow \text{tomorrow (future state)}$$

$$P = \begin{array}{c} 0 \\ 1 \end{array}\begin{bmatrix} \alpha & 1-\alpha \\ \beta & 1-\beta \end{bmatrix}$$

since rows sum to 1

today (current state)

$P(A \cap B \cap C) = P(C|A\cap B)P(B|A)P(A)$

Do 2nd

Q. Compute $P(\underset{\text{rains}}{X_0 = 0}, \underset{\text{doesn't rain}}{X_1 = 1}, \underset{\text{rains}}{X_2 = 0})$ if $P(X_0 = 0) = 0.8$.

$$P(X_0 = 0, X_1 = 1, X_2 = 0) = P(X_2 = 0 \mid X_1 = 1, X_0 = 0) \cdot \underbrace{P(X_1 = 1, X_0 = 0)}_{= P(X_1 = 1 \mid X_0 = 0) \cdot P(X_0 = 0)}$$

by Markov Property

$$= \underbrace{P(X_2 = 0 \mid X_1 = 1)}_{= P_{10} = \beta} \cdot \underbrace{P(X_1 = 1 \mid X_0 = 0)}_{P_{01} = 1-\alpha} \cdot \underbrace{P(X_0 = 0)}_{0.8}$$

assuming this is stationary transition prob.

$$= \boxed{0.8\beta(1-\alpha)}$$

Example 1: Forecasting the weather

Suppose that the chance of rain tomorrow depends on previous weather conditions only through whether or not it is raining today.

- If it rains today, then it will rain tomorrow with probability α
- If it does not rain today, then it will rain tomorrow with probability β
- Assume that the process is in state 0 when it rains and state 1 when it does not rain
- $\mathbf{P} = \begin{pmatrix} \alpha & 1-\alpha \\ \beta & 1-\beta \end{pmatrix}$

Do 1st

So, for example, say $\alpha = 0.3$ & $\beta = 0.5$

Then $P = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} 0.3 & 0.7 \\ 0.5 & 0.5 \end{bmatrix} \end{array}$

$P(A \cap B) = P(B|A)P(A)$

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Compute $P(X_0 = 0, X_1 = 1) = \underbrace{P(X_1 = 1 \mid X_0 = 0)}_{P_{01} \text{ in } P} \underbrace{P(X_0 = 0)}_{\text{suppose this is } 0.8}$

$= (0.7)(0.8) = 0.56$

Example 2: Communication system

Consider a communications system that transmits the digits 0 and 1.
Each digit transmitted must pass through several stages, at each of which there is a probability p that the digit entered will be unchanged when it leaves.

- X_n: the digit entering the nth stage $\in \{0, 1\} = S$
- There are two states.
- $\mathbf{P} = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}$ $\quad P = \begin{bmatrix} 0.1 & 0.9 \\ 0.9 & 0.1 \end{bmatrix}$

Suppose $p = 0.1$ and the initial distribution is $P(X_0 = 0) = 0.4$, $P(X_0 = 1) = 0.6$.

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Q. Determine the cond. prob. $P(X_1 = 0, X_2 = 1 \mid X_0 = 0)$.

$$P(X_1=0,\ X_2=1 \mid X_0=0) = \underbrace{P(X_2=1 \mid X_1=0,\ X_0=0)}_{\substack{\text{by MP, this} \\ \text{is } P_{01} \\ \parallel \\ 0.9}} \cdot \underbrace{P(X_1=0,\ X_0=0)}_{\substack{\parallel \\ \underbrace{P(X_1=0 \mid X_0=0)}_{\substack{P_{00} \\ \parallel \\ 0.1}} \underbrace{P(X_0=0)}_{0.4}}}$$

$$= 0.9(0.1)(0.4)$$

$$= 0.036$$

Example

Random walk model (cont'd)

A Markov chain whose state space is given by the integers $i = 0, \pm 1, \pm 2, \ldots$ is said to be a **random walk** if, for some number $0 < p < 1$,

$$P_{i,i+1} = p = 1 - P_{i,i-1}, \quad i = 0, \pm 1, \ldots$$

- State space $S = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- $p = P(i \to i+1), \quad q = P(i \to i-1)$
- Random walk if $p + q = 1$ for $0 < p < 1$
- $p = q = 1/2$: symmetric random walk
- $p \neq q$: asymmetric random walk

*

Random walk model (cont'd)

Other versions:

- Depending on state i,
  $p_i = P(i \to i+1), \quad q_i = P(i \to i-1), \quad p_i + q_i = 1$
- Allowing stay,
  $p_i = P(i \to i+1), \quad q_i = P(i \to i-1), \quad r_i = P(i \to i)$
  $p_i + q_i + r_i = 1$

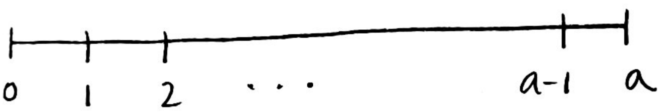
Variation where there is a possibility of staying in state i

Example

Random walk model (bounded)

Suppose that $a+1$ positions are marked out on a straight line and numbered $0, 1, \ldots, a$. A person starts at k where $0 < k < a$. The walk proceeds in such a way that at each step there is a probability p that, the walker goes 'forward' one pace to $k+1$, and a probability $q = 1-p$ that the walker goes 'back' one pace to $k-1$. The walk continues until either 0 or a is reached, and then ends.

Generally, in a random walk, the position of a walker after having moved n times is known the **state** of the walk after n **steps**. Thus the walk described above starts at state k at step 0 and moves to either state $k-1$ or state $k+1$ after 1 step, and so on.



Random walk model (cont'd)

If the walk is bounded, then the ends of the walk are known as **barriers**, and they may have various properties. In this case, the barriers are said to be **absorbing** which implies that the walk must end once a barrier is reached since there is no escape.

A simple random walk on a line occurs when a step forward $(+1)$ has probability p and a step back (-1) has probability $q\ (= 1-p)$. At the ith step, the modified Bernoulli random variable W_i is observed, and the position of the walk at the nth step is the random variable

$$X_n = X_0 + W_1 + W_2 + \cdots + W_n = X_{n-1} + W_n.$$

The random variable W_i can take the values $+1$ or -1, which is a modified Bernoulli trial in that the outcomes are $\{-1, +1\}$ rather than $\{0, 1\}$.

Example 3: Gambling Model

Consider a gambler who, at each play of the game, either (successive plays are indep.)

- wins \$1 w/ prob. p OR
- loses \$1 w/ prob. $1-p$

Suppose gambler quits playing when he/she goes broke or attains a fortune of \$N. Gambler's fortune is a MC with transition probabilities:

$$P_{i,i+1} = p \quad \text{for } i = 1, 2, \ldots, N-1$$

$$P_{i,i-1} = 1-p$$

$$P_{00} = 1$$

$$P_{NN} = 1$$

def: States 0 & N are called absorbing states since once they are entered, they are never left.

Now suppose you have finite capital i.

X_n = your fortune at time n

House: infinitely rich

$p = P(\text{win})$ & $q = 1-p = P(\text{lose})$

Ruin : You run out of capital ($X_n = 0$)

Q. If you play indefinitely, what is your chance of ruin ?

"Gambler's Ruin" Problem (§4.5)

(Alternatively, could ask : Starting with i units (e.g. \$$i$), what is the probability your fortune will reach N before reaching 0 ?)

State Space $S = \{0, 1, \ldots\}$

- If $p > q$ (i.e. $p > \frac{1}{2}$),

$$P(\text{Ruin}) = (q/p)^i$$

$$P(\text{No Ruin}) = 1 - (q/p)^i$$

$\Rightarrow$ There is a positive prob. that you will never go broke, but will instead become infinitely rich!

- If $p \leq q$ (i.e. $p \leq \frac{1}{2}$)

$$P(\text{Ruin}) = 1$$

$\Rightarrow$ With prob. 1 you will get ruined (go broke).

Why?

Condition on outcome of the initial play of the game

; see §4.5 for details!