Recall

def: A Markov chain is a sequence of RVs

$X_0, X_1, X_2, \ldots$ taking values in S (sample space) s.t.

$$P(X_{n+1} = j \mid X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n (= i))$$

$$= P(X_{n+1} = j \mid X_n = i) = P_{ij}$$

Markov property

for every $n \in \mathbb{N}$ & every sequence $i_0, i_1, \ldots, i_n \in S$ for which $P(X_0 = i_0, \ldots, X_n = i_n) > 0$.

Also,

$$P(X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n) = \alpha_{i_0} \underbrace{P_{i_0 i_1} \cdots P_{i_{n-1} i_n}}.$$

α_i's initial distribution

product of 1-step transition probabilities P_{ij}'s

Follows that

$$P(X_1 = j) = \sum_{i \in S} P(X_0 = i, X_1 = j)$$

$$= \sum_{i \in S} \alpha_i P_{ij}.$$

$* \longrightarrow$

* Chain rule for conditional probabilities :

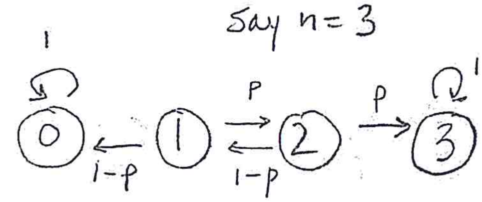
$$P(X_0 = i_0, X_1 = i_1, X_2 = i_2)$$

$$= P(X_0 = i_0)\, P(X_1 = i_1 \mid X_0 = i_0)\, P(X_2 = i_2 \mid X_0 = i_0, X_1 = i_1)$$

$$= \alpha_{i_0} P_{i_0 i_1} P_{i_1 i_2}$$

Examples

① Simple Random Walk on a finite set: (± 1 each time step)



Suppose $S = \{0, 1, \dots, n\}$

Fix $a \in S$ and let $\alpha_a = 1$ and $\alpha_i = 0$ for $i \neq a$ $(i \in S)$.

Fix $p \in \mathbb{R}$ s.t. $0 < p < 1$ and let

$$\left.\begin{aligned} P_{i,i+1} &= p \\ P_{i,i-1} &= 1-p \end{aligned}\right\} \text{ for } 0 < i < n$$

$P_{00} = P_{nn} = 1$ ← absorbing states 0 & n
(once a particle enters an absorbing state, it cannot leave it)

In matrix form

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ n-1 \\ n \end{array} \begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2 & 3 & \cdots & n-1 & n \end{array} \\ \left[\begin{array}{ccccccc} 1 & 0 & \cdots & & & & 0 \\ 1-p & 0 & p & & & & \vdots \\ 0 & 1-p & 0 & p & & & \vdots \\ & & \ddots & \ddots & \ddots & p & \\ & & & & 1-p & 0 & p \\ 0 & \cdots & & & 0 & 0 & 1 \end{array}\right] \end{array}$$

This MC starts at a & then at each step either

- increases by 1 w/prob. p
- decreases by 1 w/prob. $1-p$

until reaching state 0 or n (absorbing states).

② Unrestricted Simple Random Walk — i.e. ± 1 each time step — same setup as gambling game discussed last lec.

Let $S = \mathbb{Z}$. Fix $a \in \mathbb{Z}$ & let $\alpha_a = 1$, $\alpha_i = 0$ for $i \neq a$.

Let $P_{i,i+1} = p$ and $P_{i,i-1} = 1-p$ $\forall i \in \mathbb{Z}$, $0 < p < 1$.

This MC has no absorbing states; the particle is free to move anywhere (in increments of ± 1).

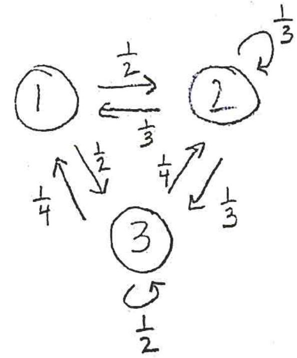
Random walk is <u>symmetric</u> if $p = \frac{1}{2}$.

e.g. $X_0 = 1$, $X_1 = 2$, $X_2 = 1$, $X_3 = 0$, $X_4 = -1$, ...

③ MC on 3 points: $\{1,2,3\}$

$S = \{1,2,3\}$

$$P = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix} \end{array}$$



Check: Sum of "out" arrows for each state is <u>1</u>.

e.g. $X_0 = 2$
$X_1 = 3$
$X_2 = 1$
$X_3 = 3$
$X_4 = 3$
⋮

4 Ehrenfest's Urn

Consider 2 urns :

- Suppose there are d balls divided between the 2 urns.
- At each step, choose 1 ball uniformly at random (from all d balls) and switch it to the opposite urn.

Let X_n = # of balls in urn 1 at time n.

Then $S = \{1, 2, \dots, d\}$ with

$$\left.\begin{aligned} P_{i,i+1} &= \frac{d-i}{d} \\ P_{i,i-1} &= \frac{i}{d} \end{aligned}\right\} \text{ for } 0 \le i \le d$$

i.e. increase # in Urn 1 by 1 : need to pick a ball from Urn 2 & switch to Urn 1

$\left(P_{i,j} = 0 \quad \text{if } j \neq i \pm 1 \right)$

One might expect that, if d is large & MC is run for a long time, that there would most likely be $\approx \frac{d}{2}$ balls in Urn 1.

* we'll consider such Q's in a bit!

Existence Theorem

<u>Thm 8.1.1</u>: Given a non-empty countable set S and non-negative numbers $\{\alpha_i\}_{i\in S}$ and $\{p_{ij}\}_{i,j\in S}$ s.t. $\sum_{i\in S} \alpha_i = 1$ and $\sum_{j\in S} p_{ij} = 1$ for each $i\in S$, $\exists$ on some prob. space $(\Omega, \mathcal{F}, P)$ a Markov chain $X_0, X_1, X_2, \ldots$ with initial probabilities α_i and transition probabilities p_{ij}.

<u>Pf</u>: See Billingsley or Rosenthal.

Let $(\Omega, \mathcal{F}, P)$ be Lebesgue measure on $[0,1]$.

Careful partitioning of $[0,1]$ & induction...

Transience, Recurrence, Irreducibility

Fundamental notions related to MCs.

For simplicity, assume $\alpha_i > 0 \ \forall i \in S$.

<u>Notation</u>: $P_i(A)$ means $P(A \mid X_0 = i)$

$E_i(A)$ means expected value w.r.t. P_i

<u>def</u>: Let $\boxed{f_{ij}^{(n)} = P_i(X_n = j, \text{ but } X_m \neq j \text{ for } 1 \leq m \leq n-1)}$

for $i,j\in S$ and $n \in \mathbb{N}$. This is the probability, starting from i, that we first hit j at time n.