def: Let $f_{ij} = P_i(\exists\, n \geq 1 : X_n = j) = \sum_{n=1}^{\infty} f_{ij}^{(n)}$.

Billingsley def $\left(= P_i\left(\bigcup_{n=1}^{\infty} \{X_n = j\} \right) \right)$ – Prob. of at least 1 visit

This is the probability, starting from i, that we ever hit state j.

def: A state $i \in S$ is recurrent (or persistent) if $f_{ii} = 1$, i.e. starting from i we are certain to return to i.

def: A state $i \in S$ is transient if it is not recurrent, i.e. if $f_{ii} < 1$.

Thm 8.2.1: Let $\{X_n : n \geq 0\}$ be a Markov chain on a state space S. Let $i \in S$. Then i is transient iff

$$P_i(X_n = i \text{ i.o.}) = 0 \quad \text{and iff} \quad \sum_{n=1}^{\infty} p_{ii}^{(n)} < \infty.$$

On the other hand, i is recurrent iff

$$P_i(X_n = i \text{ i.o.}) = 1 \quad \text{and iff} \quad \sum_{n=1}^{\infty} p_{ii}^{(n)} = \infty.$$

def: $\left[p_{ij}^{(n)} \text{ is the } n^{\text{th}} \text{ step transition probability} : P(X_{m+n} = j \mid X_m = i) \right]$

To prove this Thm, we first need a lemma:

<u>Lemma 8.2.2</u> : $P_i(\#\{n \geq 1 : X_n = j\} \geq k) = f_{ij}(f_{jj})^{k-1}$,

for $k = 1, 2, \ldots$ In words, the probability that, starting from i, we hit state j at least k times, is $f_{ij}(f_{jj})^{k-1}$.

<u>Pf</u>: Starting from i, hitting j at least k times is equiv. to first hitting j (from i) & then returning to j at least $k-1$ more times. [See B p. 117 for more details]

<u>Pf of Thm 8.2.1</u>: By continuity of prob. & by Lemma 8.2.2,

$$P_i(X_n = i \text{ i.o.}) = \lim_{k\to\infty} P_i(\#\{n \geq 1 : X_n = i\} \geq k)$$

$$= \lim_{k\to\infty} (f_{ii})^k = \begin{cases} 0 & \text{if } f_{ii} < 1 \\ 1 & \text{if } f_{ii} = 1. \end{cases}$$

This proves the first equivalence to transience & recurrence.

For the remaining 2 equivalences,

$$\sum_{n=1}^{\infty} p_{ii}^{(n)} = \sum_{n=1}^{\infty} P_i(X_n = i) \quad \text{by def}$$

$$= \sum_{n=1}^{\infty} E_i(\mathbb{1}_{X_n = i})$$

$$= E_i\left(\sum_{n=1}^{\infty} \mathbb{1}_{X_n=i}\right) \quad \text{by countable linearity}$$

$$= E_i\left(\#\{n \geq 1 : X_n = i\}\right)$$

$$= \sum_{k=1}^{\infty} P_i\left(\#\{n \geq 1 : X_n = i\} \geq k\right) \quad \text{by Prop. 4.2.9}$$

$$\left(\sum_{k=1}^{\infty} P(X \geq k) = E[\lfloor X \rfloor] \text{ for } X \text{ nonneg RV}\right)$$

$$= \sum_{k=1}^{\infty} (f_{ii})^k \quad \text{by Lemma 8.2.2}$$

$$= \begin{cases} \dfrac{f_{ii}}{1-f_{ii}} < \infty & \text{if } f_{ii} < 1 \quad (\text{geometric series}) \\ \infty & \text{if } f_{ii} = 1 \end{cases} . \quad \boxtimes$$

- Apply this Thm to simple symmetric RW ($p = \frac{1}{2}$):

  $S = \mathbb{Z}$, $X_0 = 0$

  For any $i \in \mathbb{Z}$, $P_{ii}^{(n)} = \binom{n}{n/2}\left(\frac{1}{2}\right)^n$ for n even

  half steps R & half steps L

  (and $P_{ii}^{(n)} = 0$ for n odd). Sterling's approx for $n!$

  $\Rightarrow P_{ii}^{(n)} \sim \sqrt{2/\pi n}$ for large n

  Thus, $\sum_{n=1}^{\infty} P_{ii}^{(n)} = \infty$ and all states are recurrent.

  ($\sqrt{n} \leq n$ for $n \geq 1 \Rightarrow \frac{1}{\sqrt{n}} \geq \frac{1}{n}$ & $\sum \frac{1}{n}$ diverges so $\sum \frac{1}{\sqrt{n}}$ diverges as well)

- Apply to asymmetric simple RW $(p \neq \frac{1}{2})$:

$S = \mathbb{Z}$

$P_{ii}^{(n)} = \binom{n}{n/2} p^{\frac{n}{2}} (1-p)^{\frac{n}{2}}$ with $p(1-p) < \frac{1}{4}$.

Sterling $\Rightarrow$ $P_{ii}^{(n)} \sim [4p(1-p)]^{n/2} \sqrt{2/\pi n}$ with $4p(1-p) < 1$

Thus, $\sum_{n=1}^{\infty} P_{ii}^{(n)} < \infty$ and all states are transient.

[You can fill in the details here!].

$\left[\sqrt{\frac{2}{\pi}} \sum_{n=1}^{\infty} \frac{(4p(1-p))^{\frac{n}{2}}}{\sqrt{n}} < \infty \right.$ by the ratio test (see next pg. scratch notes) $\left. \right]$

In higher dimensions, $S = \mathbb{Z}^d$ ← d-dim lattice

symmetric RW is recurrent in dimensions 1 & 2,

but transient in dimensions $d \geq 3$.

↗ counter-intuitive!

details

$$\left[\begin{array}{l} P_{ii}^{(n)} \sim \left(\sqrt{2/\pi n}\right)^d = (2/\pi n)^{d/2} \\ \sum_{n=1}^{\infty} P_{ii}^{(n)} = \infty \text{ iff } d \leq 2 \quad \to \text{p-test} \end{array} \right]$$

p-series test

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{converges if } p > 1 \\ \text{diverges if } p \leq 1 \end{cases}$$

$$\sum_{n=1}^{\infty} [\underbrace{4p(1-p)}_{a<1}]^{n/2} \sqrt{\frac{2}{\pi n}} < \infty$$

Q. Why does this converge?

$$\sqrt{\frac{2}{\pi}} \sum \underbrace{a^{\frac{n}{2}} \cdot \frac{1}{n^{\frac{1}{2}}}}$$

$$a_n = \left(\frac{a^n}{n}\right)^{\frac{1}{2}}$$

Ratio Test *

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1 \Rightarrow \sum_{n=1}^{\infty} a_n \text{ absol. convergent}$$

$$\lim_{n\to\infty} \frac{\left(\frac{a^{n+1}}{n+1}\right)^{\frac{1}{2}}}{\left(\frac{a^n}{n}\right)^{\frac{1}{2}}} = \lim_{n\to\infty} \left(\frac{\cancel{a^n} a}{n+1} \cdot \frac{n}{\cancel{a^n}}\right)^{\frac{1}{2}} = \lim_{n\to\infty} a^{\frac{1}{2}} \underbrace{\left(\frac{n}{n+1}\right)^{\frac{1}{2}}}_{\frac{\sqrt{n}}{\sqrt{n+1}}}$$

L'Hopitals Rule

$$\overset{H}{=} a^{\frac{1}{2}} \lim_{n\to\infty} \frac{\sqrt{n+1}\ \frac{1}{2} n^{-\frac{1}{2}} - \sqrt{n}\ \frac{1}{2}(n+1)^{-\frac{1}{2}}}{n+1}$$

Dim1 (62%)

To use `cutree` with `agnes` and `diana` you can perform the following:

$$\frac{\frac{1}{2}\sqrt{n+1}\cdot\frac{1}{\sqrt{n}} - \frac{1}{2}\sqrt{n}\cdot\frac{1}{\sqrt{n+1}}}{n+1}$$

```
# Cut agnes() tree into 4 groups
hc_a <- agnes(df, method = "ward")
cutree(as.hclust(hc_a), k = 4)

# Cut diana() tree into 4 groups
hc_d <- diana(df)
cutree(as.hclust(hc_d), k = 4)
```

$$= \frac{\frac{1}{2}\left(\frac{\sqrt{n+1}}{\sqrt{n}} - \frac{\sqrt{n}}{\sqrt{n+1}}\right)}{n+1}$$

$$= \frac{1}{2}\left(\frac{(\sqrt{n+1})^2 - (\sqrt{n})^2}{\sqrt{n}\sqrt{n+1}\,(n+1)}\right)$$

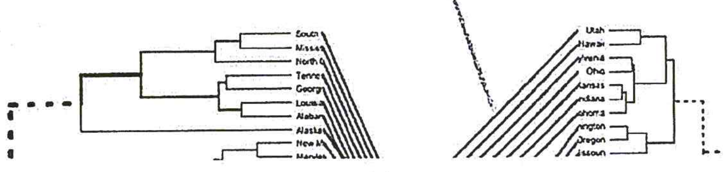
Lastly, we can also compare two dendrograms. Here we compare hierarchical clustering with complete linkage versus Ward's method. The function `tanglegram` plots two dendrograms, side by side, with their labels connected by lines.

$$= \frac{\frac{1}{2}(\cancel{n}+1-\cancel{n})}{\sqrt{n(n+1)}\,(n+1)} = \frac{1}{2\sqrt{n(n+1)}(n+1)}$$

```
# Compute distance matrix
res.dist <- dist(df, method = "euclidean")

# Compute 2 hierarchical clusterings
hc1 <- hclust(res.dist, method = "complete")
hc2 <- hclust(res.dist, method = "ward.D2")

# Create two dendrograms
dend1 <- as.dendrogram (hc1)
dend2 <- as.dendrogram (hc2)

tanglegram(dend1, dend2)
```

$$\Rightarrow \lim_{n\to\infty} \frac{1}{2\sqrt{n}\,(n+1)^{3/2}} = 0$$

$$\Rightarrow \sum_{n=1}^{\infty} a_n \text{ converges!}$$



def: A Markov chain is irreducible if $f_{ij} > 0$ $\forall i,j \in S$, i.e. if it is possible to move from any state to any other state.

Equivalently, MC is irreducible if for any $i,j \in S$, $\exists\, r \in \mathbb{N}$ s.t. $p_{ij}^{(r)} > 0$.

def: A MC is reducible if it is not irreducible, i.e. if $f_{ij} = 0$ for some $i,j \in S$.

Thm 8.2.3 : Let $P = \{p_{ij}\}_{i,j \in S}$ be the transition probability matrix for an irreducible Markov chain on a state space S. Then the following are equivalent:

(i) $\exists\, k \in S$ with $f_{kk} = 1$, i.e. k is recurrent.

(ii) $\forall\, i,j \in S$, we have $f_{ij} = 1$ → All states are recurrent.

(iii) $\exists\, k, l \in S$ with $\sum_{n=1}^{\infty} p_{kl}^{(n)} = \infty$.

(iv) $\forall\, i,j \in S$, we have $\sum_{n=1}^{\infty} p_{ij}^{(n)} = \infty$.

Pf: Clearly (ii) $\Rightarrow$ (i). We see that (i) $\Rightarrow$ (iii) follows immediately from Thm 8.2.1.

$2 \Rightarrow 1 \Rightarrow 3 \Rightarrow 4$

Next show that (iii) $\Rightarrow$ (iv):

Assume (iii) holds. Let $i, j \in S$. By irreducibility, $\exists\, m, r \in \mathbb{N}$ s.t. $P_{ik}^{(m)} > 0$ and $P_{\ell j}^{(r)} > 0$. But then, since $P_{ij}^{(m+n+r)} \geq P_{ik}^{(m)} P_{k\ell}^{(n)} P_{\ell j}^{(r)}$, it follows that

$$\sum_n P_{ij}^{(n)} \geq P_{ik}^{(m)} P_{\ell j}^{(r)} \underbrace{\sum_n P_{k\ell}^{(n)} = \infty}_{\text{from (iii)}} \quad \text{as claimed.}$$

Lastly, show that (iv) $\Rightarrow$ (ii):

[see next pg]