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2 main results give conditions under which it is true that $E[X_n] \to E[X]$ as $n \to \infty$

- Monotone Convergence Thm (R Thm 4.2.2) $E[X_1] > -\infty$ & $\{X_n\} \nearrow \{X\}$ ...
- Bounded Convergence Thm (R Thm 7.3.1) $\exists K \in \mathbb{R}$ s.t. $|X_n| \leq K \; \forall n \in \mathbb{N}$ ...

Now we will establish 2 more similar limit thms:

- Dominated Convergence Thm
- Uniformly Integrable Convergence Thm

First, another result need to prove DCT above.

<u>Thm 9.1.1 [Fatou's Lemma]</u>: If $X_n \geq C \; \forall n \in \mathbb{N}$ and some constant $C > -\infty$, then

$$E\left[\liminf_{n\to\infty} X_n\right] \leq \liminf_{n\to\infty} E[X_n]$$

(Allow possibility that both sides are ∞).

<u>Pf</u>: Let $Y_n = \inf_{k \geq n} X_k$ and $Y = \lim_{n\to\infty} Y_n = \liminf_{n\to\infty} X_n$.

SKIP

$$\left(Y_1 = \inf_{k\geq 1} X_k \;,\; Y_2 = \inf_{k \geq 2} X_k \;,\; \dots \right)$$

Then $Y_n \geq C$ (since $X_n \geq C \ \forall n$ by assumption) and $\{Y_n\} \nearrow Y$. Also, $Y_n \leq X_n$ (by defn of Y).

By order-preserving property & MCT, it follows that

$$\lim_{n\to\infty} E[Y_n] = E[Y] \quad \text{(by MCT)}$$

$$\parallel$$

$\liminf_{n\to\infty} E[Y_n]$ since the limit exists AND $E[X_n] \geq E[Y_n]$ (order-preserving)

$$\Rightarrow \liminf_{n\to\infty} E[X_n] \geq \liminf_{n\to\infty} E[Y_n] = E[Y]$$
$$= E\left[\liminf_{n\to\infty} X_n\right]$$

by def of Y. ∎

Note: $\liminf_{n\to\infty} X_n$ is interpreted pointwise

i.e. its value at ω is $\liminf_{n\to\infty} X_n(\omega)$.

* Thm 9.1.2 [Dominated Convergence Thm]: If $X_1, X_2, \ldots$ are RVs, and if $X_n \to X$ with prob. 1, and if $\exists$ a RV Y s.t. $|X_n| \leq Y \ \forall n \in \mathbb{N}$ and $E[Y] < \infty$, then

$$\lim_{n\to\infty} E[X_n] = E[X].$$

Pf: First note that $Y + X_n \geq 0$. Apply Fatou's Lemma to $\{Y + X_n\}$, we see that

(LHS is $E[X]$ if $\lim_{n\to\infty} X_n = X$ exists)

$$E[Y] + E[X] = E[Y+X] \leq \liminf_n E[Y + X_n] = E[Y] + \liminf_n E[X_n]$$

Since $E[Y] < \infty$, it follows that (cancel $E[Y]$ terms)

$$E[X] \leq \liminf_n E[X_n].$$

Similarly, $Y - X_n \geq 0$. Apply Fatou's Lem. to $\{Y - X_n\}$:

$$E[Y] - E[X] \leq E[Y] + \liminf_n E[-X_n]$$

$$= E[Y] - \limsup_n E[X_n]$$

$$\Rightarrow \quad E[X] \geq \limsup_n E[X_n].$$

However, we always have $\limsup_n E[X_n] \geq \liminf_n E[X_n]$.

Thus, combining with

$$\limsup_n E[X_n] \leq E[X] \leq \liminf_n E[X_n]$$

$$\Rightarrow \underbrace{\limsup_n E[X_n] = \liminf_n E[X_n]}_{\substack{\parallel \\ \lim_{n\to\infty} E[X_n]}} = E[X]. \quad \blacksquare$$

NOTE: If RV Y is constant, then DCT reduces to BCT.

def: A collection $\{X_n\}$ of random variables is *uniformly integrable* if $\lim_{\alpha\to\infty} \sup_n E\left[|X_n| \mathbb{1}_{|X_n|\geq\alpha}\right] = 0$.

Note: Uniform integrability $\Rightarrow$ boundedness of certain expectations

Prop 9.1.5: If $\{X_n\}$ is uniformly integrable, then $\sup_n E[|X_n|] < \infty$. Furthermore, if also $X_n \xrightarrow{a.s.} X$ as $n\to\infty$, then $E[|X|] < \infty$.

* **Thm 9.1.6 [Uniform Integrability Convergence Thm]**:

If $X_1, X_2, \ldots$ are RVs, and if $X_n \to X$ with prob. 1 as $n\to\infty$, and if $\{X_n\}$ are uniformly integrable, then

$$\lim_{n\to\infty} E[X_n] = E[X].$$

PF (Idea): Let $Y_n = |X_n - X|$ so that $Y_n \to 0$ as $n\to\infty$.

Show that $E[Y_n] \to 0$ as $n\to\infty$ by considering Y_n in 2 pieces: $Y_n = Y_n \mathbb{1}_{Y_n<\alpha} + Y_n \mathbb{1}_{Y_n\geq\alpha}$.

Then, it follows from triangle inequality that

$$|E[X_n]-E[X]| \le E[Y_n] \to 0 \quad \text{as } n\to\infty$$ which proves the theorem.

Moment Generating Functions & Large Deviations

(Ref: R §9 con't (9.3), B §9) - skip 9.2 & 9.4

(B) An interesting connection w/ SLLNs is to estimate the rate at which $\frac{S_n}{n}$ converges to the mean μ.

Proof of SLLN used upper bounds for probabilities

$$P\left(\left|\frac{S_n}{n}-\mu\right| \ge \varepsilon\right) \text{ for large } \varepsilon.$$

Accurate upper & lower bounds for these probabilities lead to the Law of the Iterated Logarithm, a theorem giving precise rates for $\frac{S_n}{n} \to \mu$.

First, we'd like to estimate the prob. of large deviations from the mean

→ requires use of moment generating functions.

Large Deviations

In particular, if $X_1, X_2, \ldots$ are i.i.d. with common mean μ and finite variance σ^2 (i.e. $E[X_i] = \mu \ \forall i$, $Var[X_i] = \sigma^2 \ \forall i$), then by Chebyshev's Ineq.

as in proof of WLLN

$$\forall \varepsilon > 0, \quad P\left(\frac{X_1 + \cdots + X_n}{n} \geq \mu + \varepsilon\right) \leq \frac{\sigma^2}{n\varepsilon} \to 0 \quad \text{as } n \to \infty$$

Q. How quickly is this limit reached?

Does the probability decrease as $O(\frac{1}{n})$, or faster?

A. If the MGFs are finite in a neighborhood of 0, then the convergence is actually exponentially fast!

Thm 9.3.4: Suppose $X_1, X_2, \ldots$ are i.i.d. with common mean μ & $M_{X_i}(s) < \infty$ for $-a < s < b$ where $a, b > 0$.

Then

$$P\left(\frac{X_1 + \cdots + X_n}{n} \geq \mu + \varepsilon\right) \leq \rho^n, \quad n \in \mathbb{N}$$

where $\rho = \inf_{0 < s < b} \left(e^{-s(\mu+\varepsilon)} M_{X_1}(s)\right) < 1.$

- This theorem gives an exponentially small upper bound on the prob. that the average of the X_i's exceeds its mean by at least ε.

- This is a simple example of a <u>large deviations</u> result.

Recall definition of <u>moment generating function</u> of a RV X:

$$M_X(s) = E[e^{sX}], \quad s \in \mathbb{R}$$

<u>Some properties</u>

- X, Y indep $\Rightarrow$ $M_{X+Y}(s) = M_X(s)\, M_Y(s)$

- $M_X(0) = 1$

- If $M_X(s) < \infty$ for $|s| < s_0$ for some $s_0 > 0$ (i.e. finite in a neighborhood of 0), then $E|X^n| < \infty$ $\forall n$ & $M_X(s)$ is <u>analytic</u> for $|s| < s_0$ with

$$M_X(s) = \sum_{n=0}^{\infty} E[X^n] s^n / n!$$

In particular, the r^{th} derivative at $s=0$ is $\overset{M_X^{(r)}(0)}{\underset{\|}{E[X^r]}}$.

Before we prove Thm 9.3.4, need a lemma.

<u>Lemma 9.3.5</u>: Let Z be a RV with $E[Z] < 0$ s.t. $M_Z(s) < \infty$ for $-a < s < b$, for some $a, b > 0$. Then

$$P(Z \geq 0) \leq \rho \quad \text{where} \quad \rho = \inf_{0 < s < b} M_Z(s) < 1.$$

<u>Pf</u>: For any $s \in (0, b)$ the function $f(x) = e^{sx}$ is increasing, so by Markov's Ineq.

$$P(Z \geq 0) = P(e^{sZ} \geq 1) \leq \frac{E[e^{sZ}]}{1} = M_Z(s)$$

Now take infimum over $0 < s < b$:

$$P(Z \geq 0) \leq \inf_{0 < s < b} M_Z(s) = \rho.$$

Lastly, since $M_Z(0) = 1$ and $M_Z'(0) = E[Z] < 0$ (by assumption), it follows that $M_Z(s) < 1$ for all positive s sufficiently close to 0, i.e. $\rho < 1$. (calculus)

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<u>Pf of Thm 9.3.4</u>: Let $Y_i = X_i - \mu - \varepsilon$ s.t. $E[Y_i] = -\varepsilon < 0$.

For $-a < s < b$, we have that

$$M_{Y_i}(s) = E[e^{sY_i}] = e^{-s(\mu+\varepsilon)} E[e^{sX_i}] = e^{-s(\mu+\varepsilon)} M_{X_i}(s) < \infty$$

(since $M_{X_i}(s) < \infty \ \forall i$ by assumption).

Then by Lemma 9.3.5,

$$P\left(\frac{X_1 + \cdots + X_n}{n} \geq \mu + \varepsilon\right) = P\left(\frac{Y_1 + \cdots + Y_n}{n} \geq 0\right)$$

$$= P\left(Y_1 + \cdots + Y_n \geq 0\right)$$

$$\leq \inf_{0<s<b} M_{Y_1 + \cdots + Y_n}(s) = \inf_{0<s<b} \left(M_{Y_1}(s)\right)^n = \rho^n$$

since Y_i's are indep

where $\rho = \inf_{0<s<b} M_{Y_1}(s) = \inf_{0<s<b} \left(e^{-s(\mu+\varepsilon)} M_{X_1}(s)\right)$.

Lastly, $\rho < 1$ by Lemma 9.3.5. $\blacksquare$