Last Time: Stated the Extension Theorem, Started working towards the proof which requires a sequence of Lemmas.

Today: Show that $\mathcal{M}$ is a σ-algebra containing $\mathcal{J}$. Do this via Lemmas (below).

Lemma 4: $\mathcal{M}$ is an algebra, i.e. $\Omega \in \mathcal{M}$ and $\mathcal{M}$ is closed under complement & finite intersection (& hence also finite $\cup$)

Pf: Recall defn of $\mathcal{M}$:

$$\mathcal{M} = \{A \subseteq \Omega : P^*(A \cap E) + P^*(A^c \cap E) = P(E) \ \forall E \subseteq \Omega\}.$$

Clearly, $\Omega \in \mathcal{M}$ and $\mathcal{M}$ is closed under complement. Remains to show $\mathcal{M}$ is closed under finite intersections. Let $A, B \in \mathcal{M}$. Then for any $E \subseteq \Omega$, using subadd.,

$$P^*((A \cap B) \cap E) + P^*((A \cap B)^c \cap E)$$

$$= P^*(A \cap B \cap E) + \underbrace{P^*((A^c \cup B^c) \cap E)}_{\overset{\shortparallel}{P^*(A^c}} \quad ?$$

$\circ\circ\circ$ DeMorgan's: $(A \cap B)^c = A^c \cup B^c$

$$= P^*(A\cap B\cap E) + P^*((A^c\cap B\cap E)\cup(A\cap B^c\cap E)\cup(A^c\cap B^c\cap E))$$

$$\le \underbrace{P^*(A\cap \underline{B\cap E}) + P^*(A^c\cap \underline{B\cap E})}_{\substack{\shortparallel \\ P^*(B\cap E) \\ \text{since } A\in\mathcal{M}}} + \underbrace{P^*(A\cap \underline{B^c\cap E}) + P^*(A^c\cap \underline{B^c\cap E})}_{\substack{\shortparallel \\ P^*(B^c\cap E) \\ \text{since } A\in\mathcal{M}}}$$

$$= P^*(B\cap E) + P^*(B^c\cap E)$$

$$= P^*(E) \quad \text{since } B\in\mathcal{M}.$$

Thus, $A\cap B\in\mathcal{M}$ and so $\mathcal{M}$ is closed under finite $\cap$.

<u>Lemma 5</u>: Let $A_1, A_2, \ldots \in\mathcal{M}$ be disjoint. For each $m\in\mathbb{N}$, let $B_m = \bigcup_{n\le m} A_n$. Then $\forall\, m\in\mathbb{N}$ and $\forall\, E\subseteq\Omega$,

$$P^*(B_m\cap E) = \sum_{n\le m} P^*(A_n\cap E)$$

<u>Pf</u>: See Rosenthal.

Lemma 5 is used in Proof of Lemma 6.

<u>Lemma 6</u>: Let $A_1, A_2, \ldots \in\mathcal{M}$ be disjoint. Then $\bigcup_n A_n\in\mathcal{M}$.

(i.e. $\mathcal{M}$ closed under <u>countable</u> $\cup$s).

when A_i's disjoint

Pf: For each $m \in \mathbb{N}$, let $B_m = \bigcup_{n \le m} A_n$.

$B_1 = A_1$

$B_2 = A_1 \cup A_2$

$B_3 = A_1 \cup A_2 \cup A_3$

$\vdots$

Then for any $m \in \mathbb{N}$ and $E \subseteq \Omega$,

$$p^*(E) = p^*(B_m \cap E) + p^*(B_m^c \cap E) \quad \text{since } B_m \in \mathcal{M}.$$

$$= \sum_{n \le m} p^*(A_n \cap E) + p^*(B_m^c \cap E) \quad \text{by Lemma 5}$$

$$\ge \sum_{n \le m} p^*(A_n \cap E) + p^*\left(\left(\bigcup_n A_n\right)^c \cap E\right)$$

where the ineq. follows by monotonicity since $\left(\bigcup_n A_n\right)^c \subseteq B_m^c$. This is true for any $m \in \mathbb{N}$, so

$$p^*(E) \ge \sum_n p^*(A_n \cap E) + p^*\left(\left(\bigcup_n A_n\right)^c \cap E\right)$$

$$\ge p^*\left(\left(\bigcup_n A_n\right) \cap E\right) + p^*\left(\left(\bigcup_n A_n\right)^c \cap E\right).$$

↑ by subadditivity

Hence, $\bigcup_n A_n \in \mathcal{M}$. ☒

Lemma 7: $\mathcal{M}$ is a σ-algebra.

Pf: Suffices to show that $\bigcup_n A_n \in \mathcal{M}$ for any sequence of subsets $A_1, A_2, \ldots \in \mathcal{M}$ (not necessarily disjoint).

Let $D_1 = A_1$ and $D_i = A_i \cap A_1^c \cap \cdots \cap A_{i-1}^c$ for $i \geq 2$.

Then $\{D_i\}$ are disjoint with $\bigcup_i D_i = \bigcup_i A_i$ & $D_i \in \mathcal{M}$ by Lemma 4. Then by Lemma 6, $\bigcup_i D_i \in \mathcal{M}$ and hence $\bigcup_i A_i \in \mathcal{M}$.

* Closed under countable $\cup$s.

Lemma 8: $\mathcal{J} \subseteq \mathcal{M}$.

Pf: Let $A \in \mathcal{J}$. Then since $\mathcal{J}$ is a semialgebra, we can write $A^c = J_1 \cup \cdots \cup J_k$ for some disjoint $J_1, \ldots, J_k \in \mathcal{J}$. Also, for any $E \subseteq \Omega$ and $\varepsilon > 0$, by defn of P^* we can find $A_1, A_2, \ldots \in \mathcal{J}$ with $E \subseteq \bigcup_n A_n$ and $\sum_n P(A_n) \leq P^*(E) + \varepsilon$.

Then $P^*(A \cap E) + P^*(A^c \cap E)$

$$\leq P^*\left(A \cap \left(\bigcup_n A_n\right)\right) + P^*\left(A^c \cap \left(\bigcup_n A_n\right)\right) \quad \text{by monotonicity}$$

$= \ldots$ (details see Rosenthal), use subadditivity & fact that $P^* = P$ on $\mathcal{J}$

Thus, we show that

$$P^*(A \cap E) + P^*(A^c \cap E) \le P^*(E) + \varepsilon.$$

This is true for any ε, hence we get

(Prop. A.3.1 : $\forall \varepsilon > 0$, if $a \le b + \varepsilon$ then $a \le b$)

$$P^*(A \cap E) + P^*(A^c \cap E) \le P^*(E).$$

By alt. defn of $\mathcal{M}$, then $A \in \mathcal{M}$. This holds $\forall A \in \mathcal{J}$ and hence $\mathcal{J} \subseteq \mathcal{M}$. $\boxtimes$

Proof of Extension Theorem is complete!
(By Lemmas 1, 3, 7, 8).

Back to the Uniform(0,1) distribution

$\Omega = [0,1]$

$\mathcal{F} = \mathcal{M}$ from previous section

Probability Measure : P^* or P or λ (all stand for Lebesgue measure)

$(\Omega, \mathcal{M}, \lambda)$ is the probability triple

Again, λ is Lebesgue measure on $[0,1]$.

$\lambda(\{x\}) = 0$ for any singleton set $\{x\} \in [0,1]$.

It follows that $\lambda(A) = 0$ for any subset $A \subseteq [0,1]$ that is countable.

e.g. Rational numbers in $[0,1]$

Integer roots of rationals in $[0,1]$

Algebraic numbers

etc.

Back to questions posed last week:

Let $X \sim U(0,1)$. Then

$P(X \text{ is rational}) = 0$

$P(X^n \text{ rational for some } n \in \mathbb{N}) = 0$

$P(X \text{ is algebraic}) = 0$