Recall defn of Random Variable: (Ref: $\mathbb{R}$ Analy. McDonald Weiss ch.7)

def: Let $(\Omega, \mathcal{F}, P)$ be a probability space. A real-valued function X on Ω is a random variable if
$$\{\omega \in \Omega : X(\omega) \in B\} \in \mathcal{F} \text{ for each Borel set } B \in \mathcal{B}.$$
aka $\left(\begin{array}{c} \{X \in B\} \in \mathcal{F} \\ X^{-1}(B) \in \mathcal{F} \end{array}\right)$

Examples:

① X is a discrete RV if $\exists$ a countable set K s.t. $P(X \in K) = 1$. We can write $K = \{x_n\}_n$.

Then the probability mass function (PMF) of X is
$$p_X : \mathbb{R} \to [0,1] \text{ defined by } p_X(x_n) = P(X = x_n).$$

Note that p_X is 0 on K^c. Also, $\mu_X(A) = \sum_{x_n \in A} p_X(x_n)$ for $A \subseteq K$.

② Suppose 2 dice are rolled. Probability space:
$$\Omega = \{(i,j) : i,j = 1,2,\dots,6\}$$
$$\mathcal{F} = \mathcal{P}(\Omega) = 2^{\Omega} \text{ (Power set)}$$
$$P = \frac{\gamma}{36} \text{ where } \gamma \text{ is counting measure}$$

Let X = sum of the 2 dice.

X is a discrete RV since $P(X \in \underbrace{\{2,3,\ldots,12\}}_{\text{countable (finite in this case) set } K}) = 1$.

PMF of X is

$$p_X(x) = \begin{cases} (x-1)/36, & x = 2,3,\ldots,7 \\ (13-x)/36, & x = 8,9,\ldots,12 \\ 0, & \text{o.w.} \end{cases}$$

(3) X is an <u>absolutely continuous</u> RV if $\exists$ a non-negative Borel measurable function f s.t.

$$\mu_X(B) = \int_B f \, d\lambda \quad \text{for all } B \in \mathcal{B}.$$

Typically write $f = f_X$ and call f_X the probability density function (PDF) of X. Also, λ = Lebesgue measure.

(4) Suppose a random number is chosen from $[0,1]$, let X denote the random # obtained. Then for $B \in \mathcal{B}$, we have

$$\mu_X(B) = P(X \in B) = \lambda(B \cap [0,1]) = \int_B \chi_{[0,1]} \, d\lambda.$$

X is an abs. continuous RV with PDF $f_X = \chi_{[0,1]}$.

characteristic fct of [0,1]

In this case, the characteristic function is the indicator function on $[0,1]$:

not to be confused with probability defn!

$$\chi_{[0,1]} = 1_{[0,1]} = \begin{cases} 1 & \text{if } x \in [0,1] \\ 0 & \text{if } x \notin [0,1] \end{cases}$$

Such a RV has the uniform distribution on $[0,1]$.

5 X is a continuous RV if

$$P(X=x) = 0 \quad \forall x \in \mathbb{R}.$$

Note: $P(X \in K) = 0$ for each countable subset $K \subset \mathbb{R}$

Note: An absolutely continuous RV is a continuous RV, but converse is NOT true.

Independence

* Informally, events or random variables are independent if they don't affect each other's probabilities.

def: Two events E & F are independent if

$$\boxed{P(E \cap F) = P(E) \cdot P(F)}$$

If E & F are not indep. they are dependent

For more than 2 events, need to carefully distinguish between pairwise independence and mutual independence.

def: Let $(\Omega, \mathcal{F}, P)$ be a probability space. Events $A_1, A_2, \dots, A_n$ are said to be mutually independent if for each subset $\{i_1, i_2, \dots, i_m\}$ of $\{1, 2, \dots, n\}$, we have

$$\boxed{P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_m}) = P(A_{i_1}) P(A_{i_2}) \cdots P(A_{i_m})}.$$

Note: Mutually indep. events are pairwise indep., but not vice versa.

def: Events $A_1, \dots, A_n$ are pairwise independent if for all $i \neq j$, A_i and A_j are independent.

More generally, the events of an arbitrary (possibly infinite) collection $\{A_i\}_{i \in I}$ are called mutually independent if every finite number of them are mutually indep.

Independent RVs

Intuitively, (consider 2 RVs) 2 RVs are indep. if Knowing the value of 1 of the variables does not affect the probability distribution of the other RV.

def: Random variables X & Y (both defined on $(\Omega, \mathcal{F}, P)$ Prob. space) are independent if, for all Borel sets A & B, the events $\{X \in A\}$ and $\{Y \in B\}$ are independent.

That is,
$$\boxed{P(X \in A, Y \in B) = P(X \in A)\, P(Y \in B)}.$$

More generally,

RVs $X_1, \dots, X_n$ defined on the same prob. space $(\Omega, \mathcal{F}, P)$ are mutually independent if

$$\boxed{P(X_1 \in B_1, \dots, X_n \in B_n) = P(X_1 \in B_1) \cdots P(X_n \in B_n)}$$

for all Borel sets $B_1, \dots, B_n$.

(As for events, the RVs of an infinite collection are mut. indep. if the RVs of each finite subcollection are mut. indep.)

Can also define <u>pairwise independence</u> for RVs:

RVs $\{X_i\}_{i\in I}$ (all defined on same prob. space) are pairwise indep. if for each $i,j \in I$, X_i and X_j are independent.

Again, mutually indep RVs $\Rightarrow$ pairwise indep RVs but converse is NOT true.

<u>Prop</u>: Let X & Y be independent RVs. Let $f,g:\mathbb{R}\to\mathbb{R}$ be Borel-measurable functions. Then the RVs $f(X)$ and $g(Y)$ are independent.

<u>Pf</u>: For Borel sets $B_1, B_2 \subseteq \mathbb{R}$,

$$P(f(X)\in B_1, g(Y)\in B_2) = P(X\in f^{-1}(B_1), Y\in g^{-1}(B_2))$$

since f & g are $\mathcal{B}$-meas!

$$= P(X\in f^{-1}(B_1))P(Y\in g^{-1}(B_2))$$

$$= P(f(X)\in B_1)\,P(g(Y)\in B_2).$$

Prop: Let X & Y be RVs defined on same prob. space $(\Omega, \mathcal{F}, P)$. Then X & Y are indep. iff

$$\boxed{P(X \le x, Y \le y) = P(X \le x) P(Y \le y)} \quad (*)$$

i.e. $F_{X,Y}(x,y) = F_X(x) F_Y(y)$

(joint CDF is product of marginal CDFs)

Pf: X & Y indep $\Rightarrow$ $(*)$ by definition.

Now suppose $(*)$ holds. Then ... see details in McDonald or Rosenthal.

Continuity of Probabilities

(Ref: § 3.3 Rosenthal, § 4 Billingsley)

Given a probability space $(\Omega, \mathcal{F}, P)$ and events $A, A_1, A_2, \ldots \in \mathcal{F}$,

$\{A_n\} \nearrow A$ means that

$$A_1 \subseteq A_2 \subseteq \cdots \text{ and } \bigcup_n A_n = A.$$

The events A_n <u>increase</u> to A.

Similarly,

$$\{A_n\} \searrow A \quad \text{means that } \{A_n^c\} \nearrow A^c$$

OR equivalently

$$A_1 \supseteq A_2 \supseteq \cdots \quad \text{and} \quad \bigcap_n A_n = A.$$

The events A_n <u>decrease</u> to A.

<u>Prop (Continuity of Probabilities)</u>: If $\{A_n\} \nearrow A$ or If $\{A_n\} \searrow A$, then $\boxed{\lim_{n\to\infty} P(A_n) = P(A)}$.

<u>Pf</u>: First suppose that $\{A_n\} \nearrow A$. Let $B_n = A_n \cap A_{n-1}^c$.

Then the $\{B_n\}$ are disjoint with

$$\bigcup_n B_n = \bigcup_n A_n = A.$$

Hence,
$$P(A) = P\Big(\underbrace{\bigcup_{m=1}^{\infty} B_m}_{\text{disjoint union}}\Big) = \sum_{m=1}^{\infty} P(B_m)$$

$$= \lim_{n\to\infty} \sum_{m=1}^{n} P(B_m) = \lim_{n\to\infty} P\Big(\underbrace{\bigcup_{m\le n} B_m}_{\text{again this is a disjoint }\cup}\Big)$$

$$\begin{array}{ccccc} B_1 & \cup & B_2 & \cup & B_3 \\ \| & & \| & & \| \\ A_1 & & A_2 \cap A_1^c & & A_3 \cap A_2^c \end{array}$$

$$= \lim_{n\to\infty} P(A_n) \quad \text{since } \{A_n\} \text{ is a nested sequence.}$$

Similarly, suppose that $\{A_n\} \searrow A$. Pf left as an exercise ☒