Last time: Stated Continuity of Probability Thm:

If $\{A_n\} \nearrow A$ or $\{A_n\} \searrow A$, then $\boxed{\lim_{n\to\infty} P(A_n) = P(A).}$

Recall definitions:

Given a probability space $(\Omega, \mathcal{F}, P)$ & events $A, A_1, A_2, \dots \in \mathcal{F}$,

(a) $\{A_n\} \nearrow A$ means that $\underbrace{A_1 \subseteq A_2 \subseteq \dots}_{\text{increasing set of events}}$ and $\bigcup_{n=1}^{\infty} A_n = A$.
"increases"

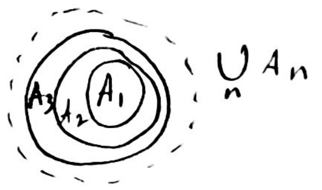
(b) $\{A_n\} \searrow A$ means that $\underbrace{A_1 \supseteq A_2 \supseteq \dots}_{\text{decreasing set of events}}$ and $\bigcap_{n=1}^{\infty} A_n = A$.
"decreases"

Alt. version of Continuity of Probability Thm:

(i) If (a) holds, then $\lim_{n\to\infty} P(A_n) = P(A)$.

(ii) If (b) holds, then $\lim_{n\to\infty} P(A_n) = P(A)$.

Pf of (i): First suppose that (a) holds. Since $A_{n-1} \subseteq A_n$, $\forall n \geq 2$ we see that $A_{n-1} \cap A_n = A_{n-1}$. Let us consider $B_n = A_n \cap A_{n-1}^c$, namely the part of A_n not in A_{n-1}, for $n \geq 2$ and $B_1 = A_1$.



$$\boxed{B_n = A_n \cap A_{n-1}^c}$$

$$B_1 = A_1$$

$$B_2 = A_2 \cap A_1^c$$

$$B_3 = A_3 \cap A_2^c$$

Notice that the $\{B_n\}$ are disjoint with

$$\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n = A \qquad \left(\text{\& also } \bigcup_{i=1}^{n} B_i = A_n \right).$$

Therefore, since P is countably additive,

$$P(A) = P\Big(\underbrace{\bigcup_{i=1}^{\infty} B_i}_{\text{disjoint } \cup \text{ of events}}\Big) = \sum_{i=1}^{\infty} P(B_i) = \lim_{n\to\infty} \sum_{i=1}^{n} P(B_i)$$

$$= \lim_{n\to\infty} P\Big(\underbrace{\bigcup_{i=1}^{n} B_i}_{\text{finite disjoint union}}\Big) = \lim_{n\to\infty} P(A_n)$$

since $\{A_n\}$ is a nested sequence.

<u>Pf of (ii)</u>: Exercise to the reader.

Limit Sets

(Ref §4, Rosenthal §3.4) Given a probability space $(\Omega, \mathcal{F}, P)$:

def: For a sequence of events $A_1, A_2, \ldots \in \mathcal{F}$, define

$$\limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k = \{A_n \text{ i.o.}\}$$

"infinitely often"

$$\liminf_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k = \{A_n \text{ a.a.}\}$$

"almost always"

These sets (events) are called the limits superior & inferior of the sequence $\{A_n\}$. ~~They lie in $\mathcal{F}$ if all the A_n's do.~~ Since $\mathcal{F}$ is a σ-algebra, $\limsup_n A_n \in \mathcal{F}$, $\liminf_n A_n \in \mathcal{F}$.

$\limsup_{n\to\infty} A_n$ – "A_n infinitely often"
- Stands for those $\omega \in \Omega$ which are in infinitely many of the A_n.
- The event that infinitely many of the events A_n occur.

$\liminf_{n\to\infty} A_n$ – "A_n almost always"
- Stands for those $\omega \in \Omega$ which are are in all but finitely many of the A_n's.

$\Omega = \{(r_1, r_2, \dots) : r_i = \overset{\text{tails}}{0} \text{ or } \overset{\text{heads}}{1}\}$ so, $H_n = \{(r_1, r_2, \dots) \in \Omega : r_n = 1\}$ (heads)

<u>Example</u>: Suppose $(\Omega, \mathcal{F}, P)$ is infinite fair coin tossing, and H_n = n^{th} coin is heads.

event

$$\limsup_n H_n = \{\text{infinitely many heads}\}$$

these are events

$$\liminf_n H_n = \{\text{all but a finite number of coin tosses are heads}\}$$

$$= \{\text{only finitely many tails}\}$$

<u>Note</u>: $\left\{\bigcap_{k=n}^{\infty} A_k\right\}$ are increasing to $\liminf_n A_n$

$\nearrow$

$\left\{\bigcup_{k=n}^{\infty} A_k\right\}$ are decreasing to $\limsup_n A_n$

$\searrow$

<u>Note</u>: If $\omega \in \Omega$ lies in all but finitely many of the A_n's, then of course it lies in infinitely many of the A_n's. Thus, always true that

$$\boxed{\liminf_n A_n \subseteq \limsup_n A_n.}$$

Another Example

(To understand this terminology)

"Infinitely many" is a consequence of "All but finitely many", but they are not the same concept.

- There are infinitely many even numbers.
- Not the case that all but finitely many $\mathbb{N}$ are even, since $1, 3, 5, \ldots$ are not even.
- All but finitely many $\mathbb{N}$ are > 12; all but $1, 2, 3, \ldots, 12$ are greater than 12.

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moved earlier [Since $\mathcal{F}$ is a σ-algebra, $\limsup_n A_n \in \mathcal{F}$ and $\liminf_n A_n \in \mathcal{F}$.

By DeMorgan's Laws:

$$\left(\limsup_n A_n\right)^c = \liminf_n (A_n^c).$$

$$\left(\left[\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k\right]^c = \cup\left(\cup A_k\right)^c = \cup\left(\cap A_k^c\right) = \liminf_n (A_n^c)\right)$$

$$\Rightarrow P(A_n \text{ i.o.}) = 1 - P(A_n^c \text{ a.a.})$$

Theorem: (i) For each sequence $\{A_n\}$,

$$P(\liminf_{n\to\infty} A_n) \le \liminf_{n\to\infty} P(A_n) \le \limsup_{n\to\infty} P(A_n) \le P(\limsup_{n\to\infty} A_n)$$

(ii) If $A_n \to A$ as $n\to\infty$, then $P(A_n) \to P(A)$ as $n\to\infty$.

Pf: (ii) clearly follows from (i) so it suffices to prove (i). For (i), the middle inequality holds by definition and the last $\le$ is similar to the first $\le$, so we will only prove the first $\le$.

Recall that events $\{\bigcap_{k=n}^{\infty} A_k\} \nearrow \liminf_{n\to\infty} A_n$.

Then by continuity of probabilities (~~[illegible]~~),

$$P(\liminf_{n\to\infty} A_n) \equiv P\left(\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} A_k\right) = \lim_{n\to\infty} P\left(\bigcap_{k=n}^{\infty} A_k\right)$$

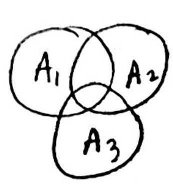
$$= \liminf_{n\to\infty} P\left(\bigcap_{k=n}^{\infty} A_k\right)$$ since if a limit exists it's equal to the liminf (and also to limsup)

$$\le \liminf_{n\to\infty} P(A_n)$$

by monotonicity. (since $\bigcap_{k=n}^{\infty} A_k \subseteq A_n$)

(increasing) nested sequence

$$\bigcap_{k=n}^{\infty} A_k \subseteq \bigcap_{k=n+1}^{\infty} A_k$$



$A_n \cap A_{n+1} \cap A_{n+2} \cap \cdots \subseteq A_{n+1} \cap A_{n+2} \cap \cdots$

$A_1 \cap A_2 \cap A_3 \subset A_2 \cap A_3$

$A_1 \cup A_2 \cup A_3 \supset A_2 \cup A_3$

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<u>Theorem</u> (The First Borel-Cantelli Lemma):

Let $A_1, A_2, \ldots \in \mathcal{F}$. If $\sum_{n=1}^{\infty} P(A_n) < \infty$, then

$$P\left(\limsup_{n\to\infty} A_n\right) = 0.$$

<u>Pf</u>: For any $m \in \mathbb{N}$, we have by countable subadditivity that

$$P\left(\limsup_{n\to\infty} A_n\right) \le P\left(\bigcup_{k=m}^{\infty} A_k\right) \le \sum_{k=m}^{\infty} P(A_k)$$

and this sum goes to 0 as $m\to\infty$ since $\sum_n P(A_n) < \infty$.

(convergent sum)

(since $\limsup_n A_n \subseteq \bigcup_{k=m}^{\infty} A_k$
& by monotonicity)

<u>Note</u>: This Thm can also be stated as:

If $\sum_{n=1}^{\infty} P(A_n) < \infty$, then $P(A_n \text{ i.o.}) = 0$.

<u>Theorem</u> (The Second Borel-Cantelli Lemma):

Let $A_1, A_2, \ldots \in \mathcal{F}$. If $\sum_{n=1}^{\infty} P(A_n) = \infty$, and $\{A_n\}$ are independent, then $P\left(\limsup_{n\to\infty} A_n\right) = 1$.

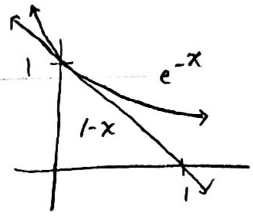
KEY ASSUMPTION!

i.e. $\liminf A_n^c$

<u>Pf</u>: Since $\left(\limsup_{n\to\infty} A_n\right)^c = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k^c$, it suffices by countable subadditivity to show that

$P\left(\bigcap_{k=n}^{\infty} A_k^c\right) = 0$ for each $n \in \mathbb{N}$. (This will show that $P\left(\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k^c\right) = 0$ and hence $P\left(\limsup_n A_n\right) = 1$.)

Now, since $1 - x \leq e^{-x}$, $(\forall x \in \mathbb{R})$



for $m, n \in \mathbb{N}$ we have that

$$
\begin{aligned}
P\left(\bigcap_{k=n}^{\infty} A_k^c\right) &\leq P\left(\bigcap_{k=n}^{n+m} A_k^c\right) = \prod_{k=n}^{n+m} P(A_k^c) \\
&= \prod_{k=n}^{n+m} \left(1 - P(A_k)\right) \quad \text{since } A_k\text{'s are independent} \\
&\leq \prod_{k=n}^{n+m} e^{-P(A_k)} \quad \text{since } 1-x \leq e^{-x}\ \forall x \in \mathbb{R} \\
&= e^{-\sum_{k=n}^{n+m} P(A_k)}
\end{aligned}
$$

HW 3 & hence A_k^c's indep.

which goes to 0 as $m \to \infty$ since $\sum_{n=1}^{\infty} P(A_n) = \infty$.

(divergent sum)

* This Theorem is striking because it asserts that if $\{A_n\}$ are independent, then

$$P(\limsup_n A_n) \text{ is either } 0 \text{ or } 1,$$

never anything in between! (e.g. $\frac{1}{2}, \frac{2}{3}$, etc.)

* We will see (next time) that this statement is true even more generally! — Kolmogorov 0-1 Law

Note: Independence assumption in 2^{nd} B-C Lemma cannot be omitted, see example below:

Consider infinite fair coin tossing & let

$$A_1 = A_2 = A_3 = \cdots = \{(r_1, r_2, \dots) \in \Omega : r_1 = 1\}$$

the event that the 1^{st} coin toss is heads

$\{A_n\}$ are NOT independent

$$P(\limsup_n A_n) = P(r_1 = 1) = \tfrac{1}{2}$$