Tail Fields

(ref: § 3.5 Rosenthal, § 4 Billingsley)

in a prob. space $(\Omega, \mathcal{F}, P)$

<u>def</u>: Given a sequence of events $A_1, A_2, \ldots \in \mathcal{F}$, we define their <u>tail field</u> by

$$\boxed{\mathcal{T} = \bigcap_{n=1}^{\infty} \sigma(A_n, A_{n+1}, \ldots)}.$$

In words, an event $A \in \mathcal{T}$ must have the property that for any $n \in \mathbb{N}$, it depends only on the events $A_n, A_{n+1}, \ldots$ In particular, $A \in \mathcal{T}$ iff changing/omitting a finite number of values does not affect the occurrence of A. (does not depend on any finite set $\{A_1, A_2, \ldots, A_n\}$)

* Tail event depends only on limiting behavior

<u>Examples</u>

- $\limsup_n A_n \in \mathcal{T}$
- $\liminf_n A_n \in \mathcal{T}$

<u>Pf</u>: See Billingsley or Rosenthal

<u>Kolmogorov's 0-1 Law</u>: If events $A_1, A_2, \ldots$ are independent with tail field $\mathcal{T}$, then for $A \in \mathcal{T}$, $P(A) = 0$ or 1.

Expectation

(ref: §5, Rosenthal §4)

Simple Random Variables

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let X be a random variable defined on this space.

def: A random variable X is simple if range(X) is finite, where range$(X) \equiv \{X(\omega) : \omega \in \Omega\}$.

assumes only finitely many values

We can list the distinct elements in the range of X as $x_1, x_2, \ldots, x_n$

and write
$$X = \sum_{i=1}^{n} x_i \mathbb{1}_{A_i}$$

where
$$A_i = \{\omega \in \Omega : X(\omega) = x_i\} = X^{-1}(\{x_i\}) \in \mathcal{F}$$

and where $\mathbb{1}_{A_i}$ are indicator functions.
$$\mathbb{1}_{A_i} = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A_i \end{cases}$$

Note that the A_i's form a finite partition of Ω.

<u>def:</u> For a simple RV $X = \sum_{i=1}^{n} x_i \mathbb{1}_{A_i}$, we define its <u>expected value</u> (or <u>expectation</u> or <u>mean</u>) by

$$E[X] = E\left[\sum_{i=1}^{n} x_i \mathbb{1}_{A_i}\right] = \sum_{i=1}^{n} x_i P(A_i)$$

where $\{A_i\}$ is a finite partition of Ω.

<u>Notation</u>: often write $\mu_X = E[X]$.

<u>Example</u>: Let $(\Omega, \mathcal{F}, P)$ be Lebesgue measure on $[0,1]$, and define simple RVs X & Y by

$$X(\omega) = \begin{cases} 5, & \text{if } \omega > 1/3 \\ 3, & \text{if } \omega \leq 1/3 \end{cases}$$

$$Y(\omega) = \begin{cases} 2, & \text{if } \omega \text{ is rational } (\mathbb{Q} \cap [0,1]) \\ 4, & \text{if } \omega = 1/\sqrt{2} \ (\approx 0.707) \\ 6, & \text{if other } \omega \leq 1/4 \quad \text{i.e. } [0, \tfrac{1}{4}] \setminus ((\mathbb{Q} \cap [0,1]) \cup \{1/\sqrt{2}\}) \\ 8, & \text{otherwise} \end{cases}$$

Compute $E[X]$ and $E[Y]$.

- Partition of $[0,1]$: $\underbrace{[0,\tfrac{1}{3})}_{A_1}, \underbrace{[\tfrac{1}{3},1]}_{A_2}$

$$E[X] = 3\,P([0,\tfrac{1}{3})) + 5\,P([\tfrac{1}{3},1])$$
$$= 3\cdot\tfrac{1}{3} + 5\cdot\tfrac{2}{3} = \tfrac{3}{3} + \tfrac{10}{3} = \tfrac{13}{3}$$

- Partition of $[0,1]$: $\underbrace{\mathbb{Q}\cap[0,1]}_{A_1}, \underbrace{\{\tfrac{1}{\sqrt{2}}\}}_{A_2}, \underbrace{[0,\tfrac{1}{4}]\setminus(\mathbb{Q}\cap[0,1])}_{A_3}, A_4$

where $A_4 = [0,1]\setminus(A_1 \cup A_2 \cup A_3) = (\tfrac{1}{4},1]\setminus((\mathbb{Q}\cap[0,1]) \cup \{\tfrac{1}{\sqrt{2}}\})$

$$E[Y] = 2\,\underset{\substack{0\\ \text{countable}}}{\cancel{P(A_1)}} + 4\,\underset{\substack{0\\ \text{singleton}}}{\cancel{P(A_2)}} + 6\,\underbrace{P(A_3)}_{\frac{1}{4}} + 8\,\underbrace{P(A_4)}_{\frac{3}{4}}$$
$$= \frac{6}{4} + \frac{24}{4} = \frac{30}{4} = \frac{15}{2}$$

Remark: $E[\underbrace{\mathbb{1}_A}_{\text{simple RV}}] = P(A)$ for all $A \in \mathcal{F}$

$E[\underbrace{c}_{\text{simple RV}}] = c \quad \forall c \in \mathbb{R}$ (constants)

$X(\omega) = c$

Expectation is Linear

Let $X = \sum_{i=1}^{n} x_i \mathbb{1}_{A_i}$ and $Y = \sum_{j=1}^{m} y_j \mathbb{1}_{B_j}$ where

$\{A_i\}$ and $\{B_j\}$ are finite partitions of Ω.

If $a, b \in \mathbb{R}$, then $\{A_i \cap B_j\}$ is again a finite partition of Ω and we have

$$E[aX + bY] = E\Big[\sum_{i,j} (ax_i + by_j)\, \mathbb{1}_{(A_i \cap B_j)}\Big]$$

$$= \sum_{i,j} (ax_i + by_j)\, P(A_i \cap B_j)$$

Q. why? indep? partition?

$$= a \sum_i x_i P(A_i) + b \sum_j y_j P(B_j)$$

$$= a\, E[X] + b\, E[Y].$$

Note: It follows that $E\Big[\sum_{i=1}^{n} x_i \mathbb{1}_{A_i}\Big] = \sum_{i=1}^{n} x_i P(A_i)$

for ANY finite collection of subsets $A_i \subseteq \Omega$, even if they do not form a partition.

[since $P(A) = \sum_i P(A \cap B_i)$ for any partition $\{B_i\}$]

Expectation is Order Preserving

If simple RVs X & Y are s.t. $\boxed{X \leq Y}$

(meaning that $X(\omega) \leq Y(\omega)$ $\forall\, \omega \in \Omega$), then

$$\boxed{E[X] \leq E[Y]}.$$

Why is this true?

$$X \leq Y \implies Y - X \geq 0$$

$$\implies E[Y-X] \geq 0$$

$$\implies E[Y] - E[X] \geq 0 \quad \text{by linearity}$$

$$\implies E[X] \leq E[Y].$$

In particular, since

$-|X| \leq X \leq |X|$, we have that $\boxed{|E[X]| \leq E[|X|]}$.

aka Generalized Triangle Inequality

Suppose X & Y are <u>independent</u> simple RVs. Then

$$\boxed{E[XY] = E[X] \cdot E[Y]}.$$

why?

$$X = \sum_{i=1}^{n} x_i \mathbb{1}_{A_i} \quad \text{and} \quad Y = \sum_{j=1}^{m} y_j \mathbb{1}_{B_j},$$

$\{A_i\}$ and $\{B_j\}$ finite partitions of Ω,

$\{x_i\}$ and $\{y_j\}$ are distinct, then

X & Y are independent iff

$$P(A_i \cap B_j) = P(A_i)\, P(B_j) \quad \forall i,j.$$

In that case,

$$\begin{aligned} E[XY] &= \sum_{i,j} x_i y_j \, P(A_i \cap B_j) \\ &= \sum_{i,j} x_i y_j \, P(A_i)\, P(B_j) \\ &= E[X]\, E[Y]. \end{aligned}$$

<u>Note</u>: This may be false if X & Y not indep!

Example: $X = \begin{cases} +1 & \text{with prob. } \frac{1}{2} \\ -1 & \text{with prob. } \frac{1}{2} \end{cases}$

e.g. fair coin flip where heads = 1 & tails = -1 (or modified Bernoulli trial)

If $Y = X$, then

$$E[X] = E[Y] = 1\left(\tfrac{1}{2}\right) + (-1)\left(\tfrac{1}{2}\right) = 0$$

BUT $E[XY] = 1$. $\left[(1)(1)\left(\tfrac{1}{2}\right) + (-1)(-1)\left(\tfrac{1}{2}\right) = 1\right]$

Example:

We may have $E[XY] = E[X]E[Y]$ even if X & Y are NOT independent.

$$X = \begin{cases} 0 & \text{w/prob } \frac{1}{3} \\ 1 & \text{"} \\ 2 & \text{"} \end{cases}$$

$Y(\omega) = 1$ if $X(\omega) = 0$ or 2

$Y(\omega) = 5$ if $X(\omega) = 1$

Compute $E[XY]$, $E[X]$, $E[Y]$ to verify claim above