STAT 706 Homework 2
SPRING 2019

Due on Thursday February 7 at the beginning of lecture.

1. Let $\Omega = \{1, 2, 3, 4\}$ and let $\mathcal{J} = \{\{1\}, \{2\}\}$. Describe explicitly the σ-algebra $\sigma(\mathcal{J})$ generated by $\mathcal{J}$.

2. Let $\mathcal{F}_1, \mathcal{F}_2, \ldots$ be a sequence of collections of subsets of Ω, such that $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$ for each n.

   (a) Suppose that each $\mathcal{F}_i$ is an algebra. Prove that $\bigcup_{i=1}^{\infty} \mathcal{F}_i$ is also an algebra.

   (b) Suppose that each $\mathcal{F}_i$ is a σ-algebra. Show (by counter-example) that $\bigcup_{i=1}^{\infty} \mathcal{F}_i$ might not be a σ-algebra.

3. Prove that there are no countably infinite σ-algebras.

4. Prove that the extension $(\Omega, \mathcal{M}, P^*)$ constructed in the proof of the extension theorem (2.3.1 in Rosenthal) must be *complete*, meaning that if $A \in \mathcal{M}$ with $P^*(A) = 0$, and if $B \subseteq A$, then $B \in \mathcal{M}$. (It then follows from monotonicity that $P^*(B) = 0$).

5. Let P and Q be two probability measures defined on the same sample space Ω and σ-algebra $\mathcal{F}$. Give an example where $P(A) = Q(A)$ for all $A \in \mathcal{F}$ with $P(A) < \frac{1}{2}$, but such that $P \neq Q$, i.e. that $P(A) \neq Q(A)$ for some $A \in \mathcal{F}$.